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Curve Fitting Using Genetic Algorithm and its Application in Craniofacial Reconstruction

Nurul Hafiza Rahamathulla and Md Yushalify Misro*

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Gelugor, Pulau Pinang, Malaysia

ABSTRACT

A best-fit curve is required to reconstruct craniofacial fracture to ensure the preciseness of the reconstructed contour. A genetic algorithm (GA) is applied to get the best-fit curve in reconstructing the fractured part. This method provides a fast decision in obtaining suitable sets of control points to interpolate both boundary regions to form a reconstructed part without any try-and-error approach that requires altering the control points several times. The optimised sets of control points with different degrees of continuity are used to develop the reconstructed part using the quintic Bézier curve to generate a smooth curve. The best-fit curvature value of the quintic Bézier curve for each degree of continuity is compared, and the curve with the lowest absolute error of curvature is chosen as the inner and outer parts of the craniofacial fracture reconstruction.

Keywords: Best-fit curve, continuity Quintic Bézier curve, craniofacial reconstruction, genetic algorithm

INTRODUCTION

A curve is a line that does not need to be straight, and it is called a curved line (Adnan et al., 2020). Curves are commonly used in the alignment of road, railway, and highway design (Eliou et al., 2014; Misro et al., 2017). Two types of curves that are commonly used in computer graphics: the Bézier curve and the B-spline curve. Bézier and B-spline curves are similar in control points and degree of the curve, but a Bézier curve does not involve knot vectors. One of the applications of B-spline curves has been hand recognition (Ma et al., 2004), while the Bézier curve is suitable for reconstructing medical images (Abdel-Aziz et al., 2021; Amorim et al., 2020).

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E-mail addresses:
nurulhafiza391@gmail.com (Nurul Hafiza Rahamathulla)
yushalify@usm.my (Md Yushalify Misro)
* Corresponding author

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Curve fitting is a process of constructing a best-fit curve to a set of data points via interpolation technique. The approximation process of the data points is very difficult. Therefore, some researchers use artificial intelligence techniques (Loucera et al., 2014). Ueda et al. (2018) used a piecewise cubic Bézier curve to approximate the point cloud boundary using the curve fitting technique. Another important aspect that needs to be considered when combining all the data points through interpolation is curve continuity.

Continuity usually accompanies curve fitting, especially when constructing a smooth curve. Ibrahim et al. (2017) used curvature continuity to connect two quintic Bézier curves into a piecewise quintic Bézier curve before applying it on the road, while Adnan et al. (2020) used continuity with a suitable degree to connect the curves and construct the outline of key and spoon. Ibrahim et al. (2017) and Misro et al. (2018) used curvature information to estimate the maximum safe speed. Curvature is the value from which a curve deviates from being a straight line. In this research, the curvature value will be used to calculate the errors in a constructed curve.

Virtual craniofacial construction using computer vision technologies is arising, but the best-fit curve is required to reconstruct the original structure. Craniofacial is a medical term related to the skull and face bones. Craniofacial studies and treatments are important for congenital malformations or facial injuries. Craniofacial reconstruction is also a basic subject in plastic surgery, Chen et al. (2017) shares various experience of craniofacial reconstruction that relate to plastic surgery.

Craniofacial reconstruction is commonly used to reconstruct facial appearance based on the analysis of skull morphology, as mentioned in Lee & Shin (2020). Many researchers have applied various approaches to construct the missing parts of the craniofacial, such as performing craniofacial reconstruction using landmark points and skull face/(tissue) skin features.

A hybrid evolutionary computing scheme containing Particle Swarm Optimization (PSO) and Differential Evolution is performed to extract the feature point and landmark count reduction by Mansour (2020). An optimal solution of curve fitting can be evaluated using various optimisation methods. PSO is one of the optimisation methods used to obtain curve fitting by generating control points for NURBS (Adi et al., 2009).

Majeed et al. (2020) constructed the missing craniofacial parts in a 3D structure using 2D CT scan contours. They also constructed the surface using a bi-cubic rational Ball surface with C^2 continuity. Craniofacial reconstruction is also developed using Radial Basis Function and bootstrap error as in Ali et al. (2020). Majeed et al. (2022) also did a comparative study using an existing technique that uses a Bezier-like function to construct craniofacial reconstruction. Moreover, Moiduddin et al. (2020) used lightweight scaffolds in craniofacial reconstruction for facial implants, which was mentioned to be useful for orthopaedic and complicated surgical procedures.

METHODOLOGY

Bézier Curve

Bézier curve is a parametric curve given by Equation 1:

$$B(t) = \sum_{i=0}^{n} b_{i,n}(t)P_i \ , \qquad t \in [0,1] \tag{1}$$

where P_i represents the control points, and $b_{i,n}(t)$ is the basic functions of Bézier known as Bernstein basis polynomials of degree n. The polynomials coefficient is in Equation 2:

$$b_{i,n}(t)=\binom{n}{i} t^i \ (1-t)^{n-i} \quad i = 0, 1, \ldots, n \tag{2}$$

and the binomial coefficient is in Equation 3:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} . \tag{3}$$

Continuity

Continuity is important and helpful in creating a smooth curve. In this research, parametric continuity will be used. Parametric continuity is the simplest way to connect two or more curves. There are three degrees for parametric continuities, namely C^0, C^1, and C^2. Each degree of continuity has its geometric interpretation (Said et al., 2021; Ammad et al., 2022).

C^0 is a position continuity, the connecting point of two curves, where the endpoint of the first curve $B_L(t)$ is connected to the first point of the second curve, $B_R(t)$. Thus, by solving $B_L(1) = B_R(0)$, the C^0 continuity condition will be satisfied.

C^1 continuity is tangent continuity at a common point, requiring the C^0 continuity condition to be satisfied first. By solving $B_L(1) = B_R(0)$, and $B_L'(1) = B_R'(0)$, continuity condition of C^1 can be satisfied.

Second order continuity at the common point is called C^2 continuity. C^0 and C^1 need to be satisfied before applying the C^2 continuity condition. C^2 continuity can be achieved when solving $B_L(1) = B_R(0)$, $B_L'(1) = B_R'(0)$, and $B_L''(1) = B_R''(0)$.

Genetic Algorithm

GA is a class of search techniques inspired by evolutionary biology. GA is chosen because of its computational robustness in the metaheuristic method. Previously, researchers have employed GA in constructing the craniofacial parts, as discussed and mentioned in the introduction. The flexibility in GA that can set suitable parameters based on a particular scenario is the main reason why GA has become a favourite among researchers. In this research, GA is used in curve fitting to reconstruct craniofacial fracture parts using the quintic Bézier curve.

Figure 1 shows the process of GA. Firstly, sets of the initial population need to be generated. Then, each population will undergo selection, crossover, and mutation to create a set of new populations, and each new population's fitness value will be calculated. The process will be repeated until all the criteria are satisfied, then GA stops, and the best solution is obtained.

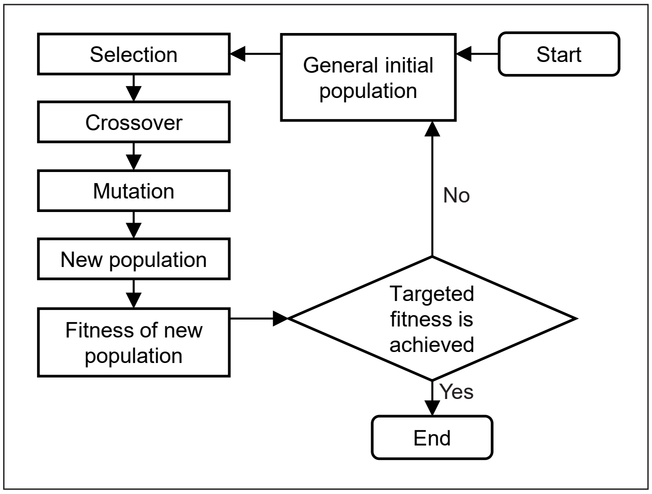


Figure 1. Flowchart of GA

Initial Population. The initial population in this research are set of coordinates to construct the fractured part. The number of coordinates or points (gene) in a solution (chromosome) depends on the degree of the Bézier curve. This research will reconstruct the fractured part using a quintic Bézier curve with degree five. Thus, a set of the solution has six coordinates $(P_0, P_1, \dots, P_5)$, while both boundary curves of the craniofacial skull use cubic Bézier curves.

Two types of the initial population used in this research are sets of C^0 and C^1 continuity solutions. These two types of curves will be used to develop the fractured part with different levels of smoothness. The set of coordinates for the initial population of C^0 continuity (P_0, P_5) and for C^1 continuity(P_0, P_1, P_4, P_5) can be determined using the idea of continuity from the previous discussion.

Therefore, the missing coordinates between these two initial populations were obtained randomly and optimised using GA, which involved four points for C^0 continuity and two points for C^1 continuity. The range of the *x*-coordinate and *y*-coordinate was set to ensure the random coordinates were still in the range set of solutions.

For the inner layer, the range of *x*-coordinates and *y*-coordinates of the control points for C^0 continuity is $x \in [152,180]$ and $y \in [155,157]$ respectively, while for C^1 continuity, the range of coordinates for the inner layer is $x \in [160,175]$ and $y \in [155,156]$. However, for the outer layer, the *x*-coordinates and *y*-coordinates of the control points for C^0 continuity are $x \in [150,190]$ and $y \in [160.5,164]$, while for curve with C^1 continuity, the range is $x \in [160,180]$ and $y \in [160.5,164]$.

Selection. Selection is a process of selecting parents to produce offspring. The total number of offspring must be equal to the original number of populations. Two solutions (parent) will be randomly selected for pairing. If there are *m* population, there will be *m/2* pairs of parents when *m* is even. If *m* is odd, *n* odd numbers of solutions with the best fitness value will advance to the next without undergoing the remaining process. Since the number of offspring is required to be equal to the population, *(m-n)*/2 pairs of parents are needed to produce the solutions. The *(m-n)*/2 pairs of parents will be selected from *n* populations chosen to advance, as Carr (2014) suggested. In this study, 10 solutions (parent) will be generated before they undergo the crossover and mutation process.

Crossover. Crossover is an important phase in GA where variations of offspring are created by exchanging the parent's gene from the crossover point. In this research, the crossover process may provide better control points to construct a best-fit curve for the fractured part compared to the original solution. Crossover points are determined randomly in this research. Figure 2 shows the crossover process, where the orange line is the crossover point. Parents 1 and 2 contained a set of six coordinates (genes) each. During the crossover, the genes are exchanged between Parents 1 and 2. There will be 5 pairs of parents in this study as 10 populations are used; thus, these pairs will undergo the crossover process.

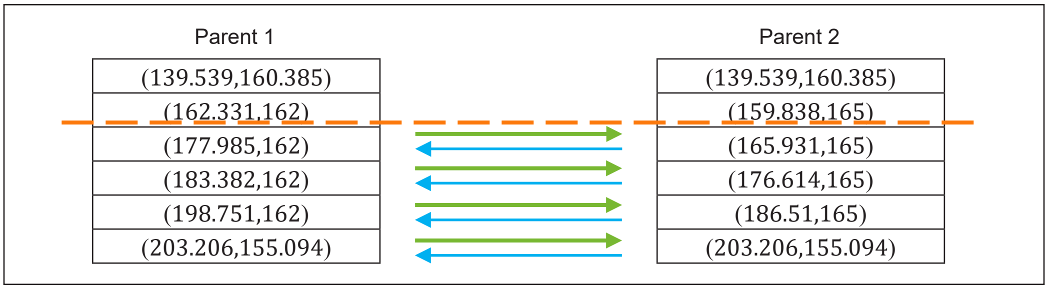


Figure 2. Crossover occurs

Mutation. The mutation is important to maintain the original population's diversity and prevent premature convergence. Premature convergence occurs when a set of coordinates (solution) with the best fitness value dominates the population. Mutation occurs in the new offspring, where two genes or coordinates will be switched. Figure 3 shows that the gene in positions two and four are switched when the mutation occurs.

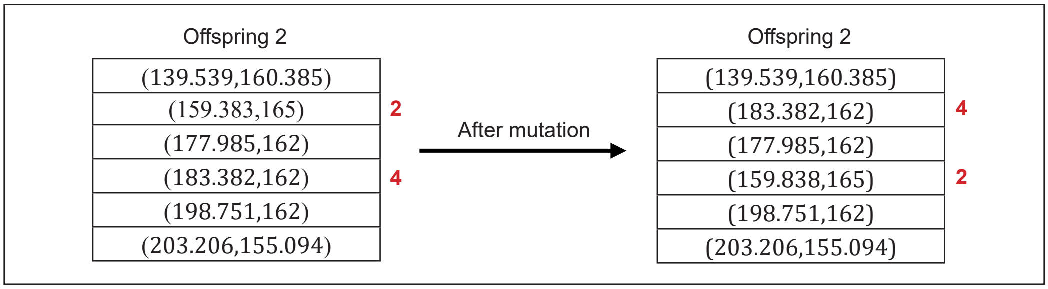


Figure 3. Mutation

Fitness Function. The fitness function is one of the crucial parts of GA. It determines the best solution compared to other solutions based on their fitness value. In this research, the fitness value of a solution will be determined using the arc length of the curve. The arc length of the curve will provide the total length of the curve. Thus, a smooth and shortest reconstructed part can be developed. Let a parametric curve $B(t) = (x(t), y(t))$ on the interval of $[a, b]$ has the length, s (Equations 4 and 5),

$$s = s(t) = \int_a^t \|B'(t)\| \, dt \tag{4}$$

where

$$\|B'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}\,. \tag{5}$$

A good solution needs a good fitness value. The lower the arc length, the better the fitness value. Therefore, the solution with the lowest arc length will reconstruct the fractured part. Hence, the constructed curve does not overfit or wiggle.

Cubic-Quintic-Cubic Piecewise Continuity Curve

The reconstruction of craniofacial structure in this research involves connected curves between cubic (A_0, A_1, A_2, A_3) – quintic (P_0, P_1, P_2, P_3, P_4 and P_5) – cubic (B_0, B_1, B_2, B_3). The cubic curves or boundary curves regarded as Curve 1 and Curve 2 were constructed using the extracted data points from the craniofacial structure of the CT scan image. For a quintic curve, three different levels of continuity will be used to construct the fractured part, which are C^0, C^1 and C^2.

The first type of curve is the quintic Bézier curve with C^0 continuity, where the curve is connected to boundary Curves 1 and 2. Thus, the control points P_0 and P_5 are the same as A_3 and B_0 such that $A_3 = P_0$ and $P_5 = B_0$. Thus, the remaining 4 control points (P_1, P_2, P_3 and P_4) of the quintic Bézier curve will be determined and optimised using GA.

The second type of curve is the quintic Bézier curve with C^1 continuity. Thus, the first and last two control points of the quintic Bezier curve, which are P_0, P_1 and P_4, P_5 will connect with A_2, A_3, and B_0, B_1, respectively. Hence, fewer control points (P_2, P_3) must be generated and optimised using GA.

The third or last type of curve is the quintic Bézier curve with C^2 continuity. C^2 continuity requires 3 control points from the first curve to connect with another curve. Since the quintic Bézier curve is an intermediate curve between both boundary curves, 3 control points from each boundary curve are enough to construct a quintic Bézier curve with 6 control points. Hence, for this type of curve, the control points do not need to be generated or optimised using GA. Therefore, this type of curve will become this study's benchmark or indicative curve.

Curvature and Surface Curvature

The curvature of the constructed curve with C^0 and C^1 continuity will be compared with the C^2 continuity curve to determine the minimum error in curve fitting. Curvature of a Bézier curve $B(t) = (x(t), y(t))$ is given by Equation 6:

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}, \tag{6}$$

where the primes are the derivative for t.

Error in curvature is computed using the absolute error of the measured and model values of curvatures. The choice of the model value varies depending on the smoothness criteria. Assuming there is no prior record of the CT scan image of a patient's skull, the C^2 continuity curve will be chosen as our model curve and the measured value for this research is the curvature value of the quintic Bézier curve with C^0 and C^1 continuities (Equation 7).

$$\text{Absolute Curvature Error} = |\text{ measured value} - \text{model value }| \tag{7}$$

RESULTS AND DISCUSSION

This research aims to obtain the best-fit curve using GA to reconstruct the craniofacial region. Figure 4 shows the original CT scan of a patient with a head injury from Majeed et al. (2015). The inner and outer layers of boundary curves are constructed as shown in Figure 5 using a cubic Bézier curve.

Normally, a CT scan is a diagnostic imaging procedure that uses a combination of X-rays and computer technology to produce images of the internal body part. Any body part, including the skull that undergoes a CT scan is a two-dimensional image representing a three-dimensional physical object. Therefore, the image of the skull can be imported into advanced computational software such as Mathematica to get the boundary points (data) of the fractured skull. Four data points from each opposite boundary are required to construct Curves 1 and 2 of the fractured part, respectively. The selected data points should produce curves aligned with the craniofacial structure on both sides, as shown in Figure 5.

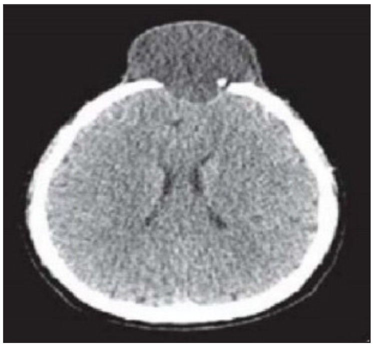


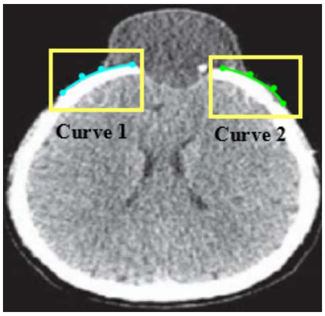
Figure 4. Original CT scan

Table 1 displays the values of the selected data points of Curves 1 and 2 of the outer and inner layers from Figure 5. Curves 1 and 2 of both layers act as boundary curves of the craniofacial region constructed using the cubic Bézier curve, where four control points are required. The data points in Table 1 are used as control points, and the curve can be obtained using Equation 1 when n is 3.

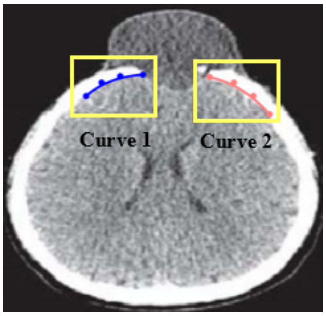
Based on Figure 5(a), Curve 1 is cyan, while the green curve is Curve 2 of the outer layer. Then, from Figure 5(b), the blue curve is Curve 1, while the pink curve is Curve 2 of the inner layer. The curves only interpolate at the endpoints from Curves 1 and 2 of both layers. Curves 1 and 2 of both layers will be used in constructing the curve of missing parts with parametric continuity.

Table 1
Data points of outer layer and inner layer of boundary curves

Points	Data points
Curve 1 of outer layer (cyan)	((102.02,143.48), (113.25,153.34), (124.27,157.68), (142.38,160.39))
Curve 2 of outer layer (green)	((194.93,158.14), (210.40,154.61), (224.38,146.34), (229.46,136.81))
Curve 1 of inner layer (blue)	((115.81,142.56), (124.76,150.57), (135.58,154.33), (148.765,155.27))
Curve 2 of inner layer (pink)	((187.3,153.76), (201.28,150.79), (212.73,142.74), (222.05,131.72))



(a)



(b)

Figure 5. Positions of data points and Curve 1 and Curve 2 of (a) outer layer and (b) inner layer

Construction of the Missing Parts

In this research, the population size is 10, and the mutation rate is 0.5. The mutation rate is 0.5, which indicate that half of the offspring will undergo the mutation process, while the crossover process will occur only at any point between point 1 and 5 with the rates of 0.6 and 0.2 for C^0 and C^1 continuity, respectively. The optimised population were constructed using the quintic Bézier curve to reconstruct the craniofacial region. The smooth connectivity of the reconstructed part will be guaranteed by imposing the continuity conditions at both ends of the reconstructed part with boundary curves. The inner and outer curves are constructed in the same way, where the only difference is their data points.

Usually, researchers that applied GA in their studies will have a fixed set value of chromosomes and a population size that only differs in chromosome combinations. This population will undergo the GA process and could provide the best solution for each generation until the fitness value is converged. However, in this research, if the set value of the chromosome is fixed, there are only four genes for C^0 continuity and two different genes for C^1 continuity. It is because some of the points will stay the same even after imposing continuity conditions. Therefore, the solution will be very limited, and the geometric continuity will not be guaranteed. This study aims to automatically generate the intermediate control points for the reconstructed curve, given the condition and limit range of the coordinates.

In this study, each population contains different sets of coordinates (chromosomes) with 6 control points for each chromosome. The remaining control points will be randomly generated and optimised using GA to complete the set of control points based on different types of continuity. Hence, the usual convergence graph is not available in this study.

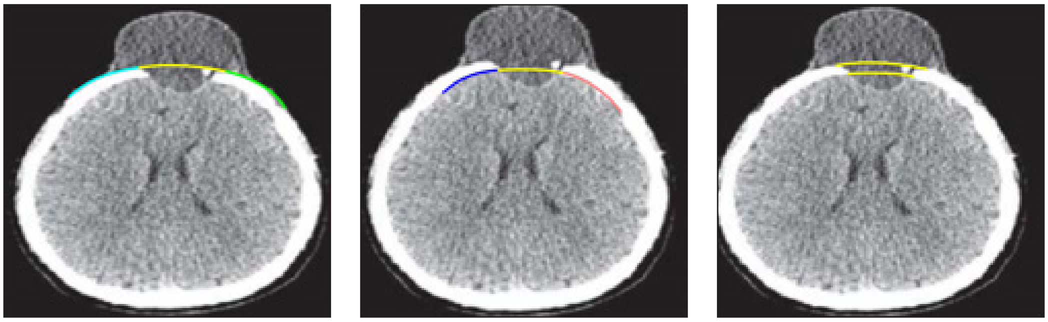
Moreover, the lowest arc length from GA will be used as stopping criteria, and the GA is stopped when the quintic Bézier curve (reconstructed part) has a similar shape and lowest arc length value with the model curve value. It applies to both C^0 and C^1 continuities. The best-fit curvature value of the quintic Bézier curve for each degree of continuity is compared, and the curve with the lowest absolute error of curvature is chosen as the inner and outer parts of the craniofacial fracture reconstruction.

Quintic Bézier curve needs to get six control points (Table 2) to connect both Curve 1 and Curve 2 with C^2 continuity. Therefore, the quintic Bézier curve with C^2 continuity will be the model curve for the fractured part of the craniofacial region. C^2 continuity contains enough information to reconstruct the smooth fracture part. Next, the quintic Bézier curve with C^0 and C^1 continuity will be compared with the model curve.

A yellow curve in Figure 6(a) and 6(b) are the outer and inner layers of the quintic Bézier curve with C^2 continuity, where the control points are taken from Table 2. Figure 6(c) shows the model curve for the

Table 2
Control points of quintic Bézier curve with C^2 continuity

Layer		Control Points	Length
Outer layer	P_0	(142.38,160.39)	52.8980
	P_1	(153.25,162.02)	
	P_2	(166.24,163.15)	
	P_3	(175.93,160.96)	
	P_4	(185.65,160.26)	
	P_5	(194.93,158.14)	
Inner layer	P_0	(148.76,155.27)	38.6597
	P_1	(156.68,155.84)	
	P_2	(165.30,155.56)	
	P_3	(169.77,155.80)	
	P_4	(178.91,155.54)	
	P_5	(187.30,153.76)	



(a) (b) (c)

Figure 6. Quintic Bézier curve with C^2 continuity for (a) outer layer, (b) inner layer and (c) both layers

inner and outer layers of craniofacial, where the reconstructed part can smoothly reattach the fractured part.

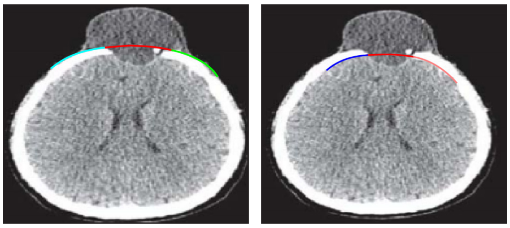
Now, the quintic Bézier curve for the inner layer and outer layer is constructed with C^0 and C^1 continuity using GA where the optimisation occurs, while the quintic Bézier curve with C^2 continuity was constructed without any optimisation taking place.

Best Fit Curve Using Genetic Algorithm (GA)

A best-fit curve with C^0 continuity and C^1 continuity was acquired using GA for inner and outer curves. By implementing this technique, medical practitioners or surgeons do not have any obligation to manually figure out the exact position (data points) to patch the broken part, as GA will automatically provide the best set of coordinates for the quintic Bézier curve that has the lowest arc length compared to another set of coordinates.

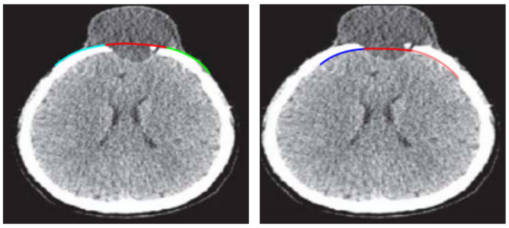
The curve with C^0 continuity and C^1 continuity evaluated using GA will be compared with the model curve, the quintic Bézier curve with C^2 continuity from Figure 6(c) for the outer and inner layers. Tables 3 and 4 show the control points of the quintic Bézier curve for the outer and inner curves and their curve length with C^0 and C^1 continuity obtained using GA, respectively. The red curve in Figures 7 and 8 are quintic Bézier curves with C^0 and C^1 continuity constructed using the control points from Tables 3 and 4, respectively.

In Figure 9, quintic Bézier curves with C^0 and C^1 continuity (red curve) are compared with the model curve (yellow curve). Based on Figure 9, there is not much difference in



(a) (b)

Figure 7. Quintic Bézier curve with C^0 using GA for: (a) outer and (b) inner layers



(a) (b)

Figure 8. Quintic Bézier curve with C^1 using GA for: (a) outer and (b) inner layers

Table 3
Control points and length of quintic Bézier curve with C^0 using GA

Outer Layer	Control points	((142.38,160.39), (170.07,162.60), (162.69,162.36), (163.88,160.52), (173.69,163.73), (194.93,158.14))
	Length	52.9327
Inner layer	Control points	((148.77,155.27), (154.29,155.65), (152.30,155.05), (163.06,156.39), (171.25,156.58), (187.30,153.76))
	Length	38.6781

Table 4
Control points and length of quintic Bézier curve with C^1 using GA

Outer Layer	Control points	((142.38,160.39), (163.18,162.54), (153.25,162.02), (178.06,161.01), (185.65,160.26), (194.93,158.14))
	Length	52.8443
Inner layer	Control points	((148.77,155.27), (156.68,155.84), (162.05,155.11), (178.91,155.54), (172.22,155.34), (187.30,153.76))
	Length	38.6218

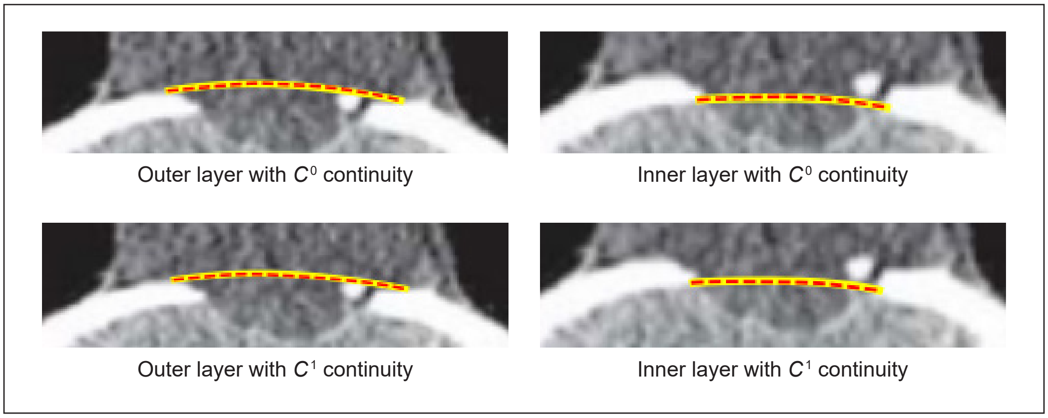


Figure 9. Overlapping of quintic Bézier curve with C^0 and C^1 continuity using GA (red) and model curve (yellow) of outer and inner layer

overlapping the red and yellow curves in both layers with C^0 and C^1 continuity. It shows that the control points obtained from GA provide a good fit curve, which is highly identical to the model curve.

Comparison of Errors in Curvature Between C^0, C^1 and C^2 Continuity

The curvature value of the best-fit curve with C^0 and C^1 continuity for both layers will be calculated. Then, the curvature error for each curve will be calculated and compared with the model curve (C^2) to determine which continuity gives the best curve for both layers.

Figures 10 and 11 show the curvature plot of the quintic Bézier curve with C^0, C^1 and C^2 continuity of outer and inner layers, respectively. The quintic Bézier curve with C^2 continuity (purple curve) in Figures 10 and 11 will be the indicator curve or benchmark curve used in this research, as the quintic Bézier curve with C^2 continuity preserves the geometric properties that are derived from boundary curves on both sides.

The curvature values from Tables 5 and 6 were used to calculate the absolute error at each t value using Equation 6. Based on Table 5, the highest error for the outer layer curve is between C^0 and C^2 continuity, where $t = 1.0$, while the lowest is between C^1 and

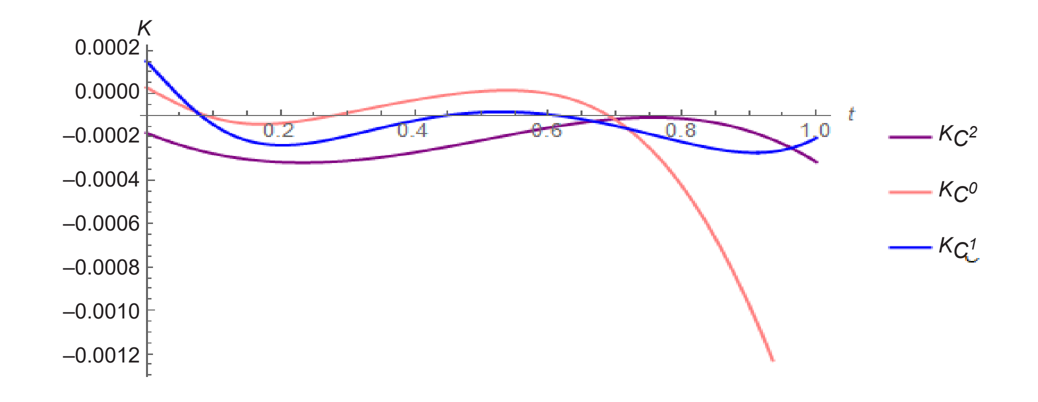


Figure 10. Quintic Bézier curve with C^0 and C^1 using GA for outer layers

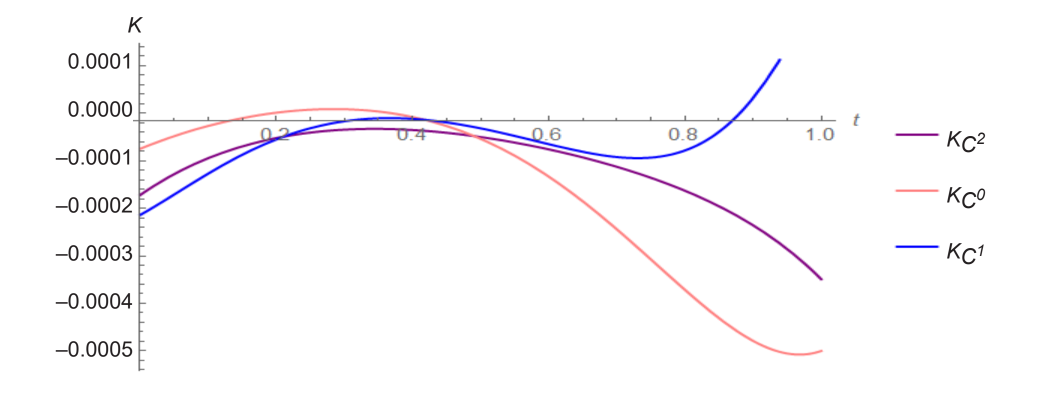


Figure 11. Quintic Bézier curve with C^0 and C^1 using GA for inner layers .

C^2 continuity, where t = 0.6. For the inner layer, the highest and lowest errors are between C^1 and C^2 continuity, where t = 1.0 and t = 0.2, respectively.

Based on Table 5, the average absolute error in curvature between the quintic Bézier curve with C^2 and C^1 is less than that between the quintic Bézier curve with C^2 and C^0 for the outer layer. Meanwhile, in Table 6, the average absolute error in curvature between the quintic Bézier curve with C^2 and C^0 is less than that between the quintic Bézier curve with C^2 and C^1 for the inner layer. Thus, the quintic Bézier curve with C^1 continuity from Figure 8 is the best-fit curve for the outer layer, while for the inner layer, the quintic Bézier curve with C^0 continuity from Figure 7 is the best-fit curve for the fractured part in the craniofacial region. Hence, this research proves that the GA is useful in determining the best-fit curve for the inner and outer layers of the craniofacial reconstruction. A lower degree of continuity can mimic the same features as C^2 and preserve the curvature continuity at both boundary curves.

Table 5
Curvature value of quintic Bézier curve with C^0, C^1 and C^2 continuity and the absolute error of outer layer

t	Curvature value of quintic Bézier curve			Absolute error	
	C^0	C^1	C^2	$\|\| C^2 - C^0 \|\|$	$\|\| C^2 - C^1 \|\|$
0	2.855×10^{-5}	1.4980×10^{-4}	-1.8064×10^{-4}	2.092×10^{-4}	3.304×10^{-4}
0.2	-1.3702×10^{-4}	-2.3745×10^{-4}	-3.1578×10^{-4}	1.788×10^{-4}	7.833×10^{-5}
0.4	-2.896×10^{-5}	-1.2451×10^{-4}	-2.7122×10^{-4}	4.423×10^{-4}	1.467×10^{-4}
0.6	5.83352×10^{-7}	-9.591×10^{-5}	-1.5734×10^{-4}	1.579×10^{-4}	6.143×10^{-5}
0.8	-4.2754×10^{-4}	-2.2279×10^{-4}	-1.1487×10^{-4}	3.127×10^{-4}	1.079×10^{-4}
1.0	-1.77593×10^{-3}	-2.0421×10^{-4}	-3.1519×10^{-4}	1.461×10^{-3}	1.110×10^{-4}
			Average	4.603×10^{-4}	1.393×10^{-4}

Table 6
Curvature value of quintic Bézier curve with C^0, C^1 and C^2 continuity and absolute error of inner layer

t	Curvature value of quintic Bézier curve			Absolute error	
	C^0	C^1	C^2	$\|\| C^2 - C^0 \|\|$	$\|\| C^2 - C^1 \|\|$
0	-7.488×10^{-5}	-2.1409×10^{-4}	-1.7226×10^{-4}	9.738×10^{-5}	4.183×10^{-5}
0.2	1.13496×10^{-6}	-5.498×10^{-5}	-5.211×10^{-5}	5.0975×10^{-5}	2.870×10^{-6}
0.4	-7.910×10^{-6}	-1.180×10^{-5}	-3.527×10^{-5}	2.736×10^{-5}	2.347×10^{-5}
0.6	-1.333×10^{-4}	-6.502×10^{-5}	-7.597×10^{-5}	5.733×10^{-5}	1.095×10^{-5}
0.8	-3.717×10^{-4}	-7.829×10^{-5}	-1.6303×10^{-4}	2.0867×10^{-4}	8.474×10^{-5}
1.0	-5.002×10^{-4}	2.7044×10^{-4}	-3.4918×10^{-4}	1.5102×10^{-4}	6.1962×10^{-4}
			Average	9.879×10^{-5}	1.3058×10^{-4}

CONCLUSION

This paper aims to present a new method that helps to develop fractured craniofacial parts using different types of continuity by matching the boundary curves (cubic Bézier curve) with the constructed curve (quintic Bézier curve). This method can be extended by applying a higher degree of curve. After satisfying the continuity conditions, the remaining control points can be calculated using GA with the lowest arc length as the fitness function while preserving the curvature continuity at both ends.

On the other hand, the size of the fractured region is an important factor in deciding the suitable degree of the curve in reconstructing the fractured part. In this study, the fractured region is considered small. Thus, C^0 and C^1 are enough to reconstruct the craniofacial region. Hence, the results are compared with C^2 continuity to validate the finding. GA is used to find the remaining control points after it undergoes a different order of continuity since moving or changing the control points will be time-consuming.

For future works, the surface of the fractured part in the craniofacial region could be reconstructed if CT scan images from different heights are available. Thus, the surface of the fractured part can be precisely constructed, and the curve can be combined to form a surface patch. Furthermore, a 3D surface can be used as a template fitting in craniofacial reconstruction is suggested to be investigated further.

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